Randall-Sundrum scenario, higher derivatives, and stability

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We demonstrate the classical stability of the weak or Planck hierarchy within the Randall-Sundrum scenario, incorporating the Goldberger-Wise, mechanism and higher-derivative interactions in a systematic perturbative expansion. Such higher-derivative interactions are expected if the RS model is the low-energy description of some more fundamental theory. Generically, higher derivatives lead to ill-defined singularities in the vicinity of effective field theory branes. These are carefully treated by the methods of classical renormalization.

I. INTRODUCTION

Theories with extra dimensions provide new ways of explaining the weak-Planck hierarchy. The original proposal for doing so appeared in Ref. [1]. An alternative proposal is the Randall-Sundrum (RS1) scenario [2], where the hierarchy is set by a relative warp factor, $e^{-k\pi r_c}$, between a "visible" brane, to which the standard model is confined, and a "hidden" brane where 4D gravity is highly localized by the RS2 mechanism [3]. Here, k is a fundamental scale determined by the 5D cosmological constant and r_c is the compactification radius. The Goldberger-Wise mechanism [4] provides a simple and natural means of stabilizing the radius at $kr_c \sim \mathcal{O}(1/\epsilon)$, by introducing a bulk scalar field with 5D mass-squared of order ϵ in fundamental units. The large observed weak/Planck hierarchy, $e^{-k\pi r_c} \sim 10^{-15}$, is then generated from a modest fundamental hierarchy, $\epsilon \sim 1/10$.

Since the RS1 field theory, including general relativity, is quantum-mechanically non-renormalizable, the model must be considered to be an effective description of a more fundamental theory. References [5] have discussed string theory embeddings of the RS1 mechanism. In any such embedding, higher-derivative interactions (α' corrections in string theory) are expected to appear in the effective field theory after integrating out very massive physics. It is, therefore, important to demonstrate that RS1 and the Goldberger-Wise mechanism are stable under the addition of such higher-derivative terms. In this paper, we will show that this is indeed the case within the systematic framework of classical effective field theory.

While short-distance quantum effects can be parametrized and studied within a local derivative expansion, it is also important to demonstrate stability in the presence of genuine, long-distance quantum effects. These can also be studied within effective field theory. References [6] have examined such effects at one loop. We hope to give a more complete treatment of quantum effects in future work.

Recently, a dual picture of the RS scenario has been developed [7], based on the AdS/CFT correspondence [8], in which the extra-dimensional dynamics is replaced by a strongly coupled conformal field theory. While this duality is compelling and powerful, we will not make use of it in this paper as some aspects remain unproven.

Our strategy is to first set up a systematic perturbative expansion for the classical effective field theory in which to study higher derivatives. While the Goldberger-Wise ϵ is an obvious small parameter of the scenario, we cannot perturbatively expand in it since the hierarchy is set by $e^{-1/\epsilon}$, which vanishes to all orders in ϵ. Instead, we choose both the bulk curvature and the brane tadpole couplings which give the Goldberger-Wise scalar a non-trivial profile in the extra dimension to provide our formal expansion parameter, λ. In Sec. II, we re-derive the Goldberger-Wise mechanism in the absence of higher derivatives. We note that there is an elegant, exactly soluble version of the Goldberger-Wise mechanism [9], but our perturbative treatment will be more convenient when higher-derivative terms are added. A discussion of the Goldberger-Wise mechanism related to ours is Ref. [10]. In Sec. III, we discuss how higher-derivative terms are constrained by symmetries. In Sec. IV we discuss the apparent incompatibility of the derivative expansion, normally valid at long distances, with the presence of "thin" or δ-function branes. We show how the ill-defined singularities that arise in the equations of motion can be eliminated by classical renormalization. In Sec. V we demonstrate the stability of the RS and Goldberger-Wise mechanisms when higher-derivative perturbations are included. The central technical concern is that terms in our perturbative expansion take the form $\lambda^n f_n(\epsilon)$ and it is important for a controlled expansion that the small parameter λ is not overwhelmed by possible large terms in f_n, such as $e^{1/\epsilon}$ or $1/\epsilon^m$. This is carefully checked. Section VI provides our conclusions.

II. THE GOLDBERGER-WISE MECHANISM IN PERTURBATION THEORY

A. The model

RS1 has a single extra dimension which is an interval, realized as an orbifold, $S^1/\mathbf{Z}_2$. We will begin by using a conventional angular coordinate, $-\pi \leq \phi \leq \pi$, for the S^1, where the orbifold symmetry acts by $\phi \to -\phi$. We will always describe fields within the fundamental domain $0 \leq \phi \leq \pi$. Their extension to general ϕ is then determined by the orbifold symmetry and periodicity on S^1.

The Goldberger-Wise mechanism will first be implemented within a theory given by

$$S = S_{bulk} + S_{vis} + S_{hid}, \qquad (2.1)$$

where

$$S_{bulk} = M^3 \int d^4x \int_{-\pi}^{\pi} d\phi \sqrt{G} \left(-\frac{1}{4} R + \frac{1}{2} (D\chi)^2 \right.$$

$$\left. + 3k^2 - \frac{1}{2} \epsilon k^2 \chi^2 \right) \tag{2.2}$$

$$S_{vis} = -M^3 k \int d^4x \sqrt{g_v} \left(\rho_v + \lambda_v \chi + \frac{1}{2} \mu_v \chi^2 \right) \tag{2.3}$$

$$S_{hid} = -M^3 k \int d^4x \sqrt{g_h} \left(\rho_h + \lambda_h \chi + \frac{1}{2} \mu_h \chi^2 \right), \tag{2.4}$$

with

$$g_{\mu\nu}^v(x) \equiv G_{\mu\nu}(x, \phi = \pi)$$

$$g_{\mu\nu}^h(x) \equiv G_{\mu\nu}(x, \phi = 0). \tag{2.5}$$

Note that we have chosen a normalization such that χ is dimensionless. We assume that there are no extremely large hierarchies among the couplings of the model. However, we do take

$$k < M$$

$$|\lambda_{v,h}| < \epsilon < |\rho_{v,h}| \sim \mu_{v,h} \sim 1. \tag{2.6}$$

We will restrict our attention to classical solutions which admit a four-dimensional Poincaré invariance. Such configurations satisfy the ansatz

$$ds^2 = e^{2A(\phi)} \eta_{\mu\nu} dx^\mu dx^\nu - r_c^2 d\phi^2$$

$$\chi = \chi(\phi), \tag{2.7}$$

where r_c is the (constant in this ansatz) "radius." In solving the equations of motion it is convenient to work with a re-scaled dimensionless extra-dimensional coordinate,

$$y \equiv k r_c \phi, \tag{2.8}$$

so that the infinitesimal distance in the extra dimension is dy/k. The Poincare ansatz then reads

$$ds^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2/k^2$$

$$\chi = \chi(y). \tag{2.9}$$

The equations of motion subject to this ansatz are

$$\chi'' + 4A'\chi' - \epsilon\chi = (\lambda_v + \mu_v\chi)\delta(y - y_c) + (\lambda_h + \mu_h\chi)\delta(y) \tag{2.10}$$

$$A'' + \frac{2}{3}(\chi')^2 = -\frac{2}{3}\left(\rho_v + \lambda_v\chi + \frac{1}{2}\mu_v\chi^2\right)\delta(y - y_c)$$

$$-\frac{2}{3}\left(\rho_h + \lambda_h\chi + \frac{1}{2}\mu_h\chi^2\right)\delta(y) \tag{2.11}$$

$$(A')^2 = 1 - \frac{1}{6}\epsilon\chi^2 + \frac{1}{6}(\chi')^2, \tag{2.12}$$

where

$$y_c \equiv k r_c \pi. \tag{2.13}$$

The brane-localized scalar tadpoles provide us with our formal small expansion parameters, $\lambda_{v,h} \sim \mathcal{O}(\lambda)$. We expand our solution as a perturbation series in λ:

$$A = \sum_n A_n$$

$$\chi = \sum_n \chi_n, \tag{2.14}$$

where the subscript n denotes the term of order λ^n in the series. Notice that at order λ^0 we have $\chi = 0$ and the (truncated) AdS$_5$ gravity solution obtained in RS1, $A_0 = -y$, with r_c arbitrary. Thus, in fluctuations away from 4D Poincaré invariance, r_c becomes a "radion" modulus at zeroth order. We will demonstrate the Goldberger-Wise mechanism for stabilizing the radius in higher orders of perturbation theory.

The strategy for solving the equations of motion is as follows. We solve for A' in terms of χ using Eq. (2.12),

$$A' = -\left[1 - \frac{1}{6}\epsilon\chi^2 + \frac{1}{6}(\chi')^2\right]^{1/2}, \tag{2.15}$$

and eliminate A' from Eq. (2.10) to obtain an equation purely for χ,

$$\chi'' - 4\chi' \sqrt{1 - \frac{1}{6}\epsilon\chi^2 + \frac{1}{6}(\chi')^2} - \epsilon\chi$$

$$= (\lambda_v + \mu_v\chi)\delta(y - y_c) + (\lambda_h + \mu_h\chi)\delta(y). \tag{2.16}$$

We will solve Eq. (2.16) to any desired order in λ. We then integrate Eq. (2.15) to solve for $A(y)$, subject to the canonical gauge choice $A(0) = 0$. Equation (2.11) will then be automatically solved away from the branes as a consequence of 5D general covariance.[1] Finally, we satisfy the two brane junction conditions of Eq. (2.11) by fine-tuning the hidden brane tension parameter ρ_h (equivalent to fine-tuning the 4D cosmological constant to zero), and adjusting the compactification radius, r_c (or equivalently, y_c) to its stable vacuum value.

B. Perturbation theory

Here, we will show self-consistently that the solution of the equations of motion satisfies

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[1] One can easily check that up to δ-function terms, Eqs. (2.10) and (2.12) imply Eq. (2.11).

$$y_c \sim \mathcal{O}(1/\epsilon)$$

$$A'(y) \approx -1, \qquad (2.17)$$

so that the weak-Planck hierarchy determined by the RS1 mechanism is set by $e^{-\mathcal{O}(1/\epsilon)}$. Therefore, while we need ϵ to be somewhat small, we cannot work perturbatively in ϵ. (As stated above we will be working strictly perturbatively in λ.) However, we will simplify our analysis by dropping subleading terms in $e^{-\mathcal{O}(1/\epsilon)}$. Note that while λ is formally small, combinations such as $\lambda \mathcal{O}(1/\epsilon^m)$ or $\lambda \mathcal{O}(e^{+1/\epsilon})$ may be large when $\epsilon \ll 1$. It is, therefore, crucial that such combinations do not appear in the perturbative series in order for the perturbative expansion to be under control. We show that this danger does not eventuate by a careful analysis.

1. Stabilization at second order

Since χ vanishes at zeroth order, we can perturbatively expand the square-root in Eq. (2.16). At first order,

$$\chi_1'' - 4\chi_1' - \epsilon\chi_1 = (\lambda_v + \mu_v\chi_1)\delta(y - y_c) + (\lambda_h + \mu_h\chi_1)\delta(y). \qquad (2.18)$$

With orbifold boundary conditions the solution is

$$\chi_1 = c_1 e^{\Delta_+(y-y_c)} + c_2 e^{\Delta_- y}, \qquad (2.19)$$

with

$$c_1 \approx \frac{-\lambda_v}{(2\Delta_+ + \mu_v)} - \frac{\lambda_h(2\Delta_- + \mu_v)e^{\Delta_- y_c}}{(2\Delta_+ + \mu_v)(2\Delta_- - \mu_h)} \qquad (2.20)$$

$$c_2 \approx \frac{\lambda_h}{(2\Delta_- - \mu_h)}, \qquad (2.21)$$

and where

$$\Delta_\pm = 2 \pm \sqrt{4 + \epsilon}. \qquad (2.22)$$

Note that for small ϵ,

$$\Delta_+ \approx 4, \quad \Delta_- \approx -\epsilon/4. \qquad (2.23)$$

In the expressions for the coefficients we have neglected subleading powers of $e^{-\Delta_+ y_c}$, since we will demonstrate Eq. (2.17).

At first order in λ the scalar field does not stabilize the radion because A receives no correction at this order. There are still two fine tunings needed to satisfy the equations of motion as in the original version of RS1 without stabilization. It is only at second order, where the first order scalar profile back reacts on the metric, that the compactification radius is fixed. This back reaction was also discussed in Ref. [10]. At second order $\chi_2 = 0$ and A_2' is given by Eq. (2.12) expanded to second order in χ. The junction conditions of Eq. (2.11) at this order read

$$\left(-1 + \frac{1}{12}\epsilon\chi_1(0)^2 - \frac{1}{12}\chi_1'(0)^2 \right)$$

$$= -\frac{1}{3}\left(\rho_h + \lambda_h\chi_1(0) + \frac{1}{2}\mu_h\chi_1(0)^2 \right) \quad (2.24)$$

$$\left(-1 + \frac{1}{12}\epsilon\chi_1(y_c)^2 - \frac{1}{12}\chi_1'(y_c)^2 \right)$$

$$= \frac{1}{3}\left(\rho_v + \lambda_v\chi_1(y_c) + \frac{1}{2}\mu_v\chi_1(y_c)^2 \right). \quad (2.25)$$

Note that parametrically in λ, the only way the visible junction condition can be solved is if

$$\delta\rho_v \equiv 3 + \rho_v \sim \mathcal{O}(\lambda^2). \qquad (2.26)$$

Such a condition will not reappear at higher orders in λ. Once we grant that ρ_v is somewhere in this $\mathcal{O}(\lambda^2)$-sized window about -3, the visible junction condition can be satisfied, not by fine tuning of couplings, but by solving for the stable vacuum value of the dynamical radius,

$$y_c = \frac{\ln(\Sigma)}{\Delta_-}, \qquad (2.27)$$

where

$$\Sigma = \frac{\Delta_+(2\Delta_- - \mu_h)(2\Delta_- - 2 - \mu_v)\lambda_v}{(\Delta_+ - \Delta_-)(4\Delta_-\Delta_+ + \mu_v(2 + \mu_v))\lambda_h}$$

$$\pm \frac{(\mu_h - 2\Delta_-)(\mu_v + 2\Delta_+)\sqrt{\left[2\delta\rho_v(4\Delta_-\Delta_+ + \mu_v(2 + \mu_v)) + \left(1 - \Delta_-\Delta_+ + \frac{\mu_v}{2}\right)\lambda_v^2\right]}}{(\Delta_+ - \Delta_-)(4\Delta_-\Delta_+ + \mu_v(2 + \mu_v))\lambda_h}. \qquad (2.28)$$

The sign will depend on the actual value of the parameters with the requirement that y_c is real and positive. It follows that in fluctuations away from 4D Poincaré invariance, the associated radion has acquired a mass-squared at this order

in λ. For a large range of the parameters, this mass-squared is positive. For example, if $\mu_{v,h}$ dominate over $\Delta_\pm$ in Eq. (2.16), then in this limit the computation of the radion effective potential at second order is precisely the one performed

by Goldberger and Wise, with the identification of their $v_{v,h}$ with our $-\lambda_{v,h}/\mu_{v,h}$. A positive mass-squared results [4,11,10].

The hidden junction condition at $y=0$ gives us a fine-tuning condition for ρ_h

$$\rho_h = 3 + \frac{(2\Delta_-(-2+\Delta_-\Delta_+)+\mu_h)\lambda_h^2}{4(\mu_h-2\Delta_-)}. \qquad (2.29)$$

This fine-tuning which we must do is equivalent to fine-tuning the effective four-dimensional cosmological constant to zero in order to permit solutions with 4D Poincaré invariance. We must perform such a fine-tuning order by order in λ.

Let us now check that our basic claims (2.17) are satisfied at this order. For generic values of the couplings,

$$\ln\Sigma \sim \mathcal{O}(1), \qquad (2.30)$$

so by Eq. (2.23), $y_c \sim \mathcal{O}(1/\epsilon)$. Given the explicit form for χ_1 it is straightforward to see that the resulting A_2 satisfies, $A' \approx -1$. We will show that these successes are maintained at higher order in λ.

2. Subdominance of higher orders

At higher order $n \geq 2$, after expanding the square-root in Eq. (2.16), we must solve the following equation for χ:

$$\chi_n'' - 4\chi_n' - \epsilon\chi_n - \mu_v\chi_n\delta(y-y_c) - \mu_h\chi_n\delta(y) = J_n, \qquad (2.31)$$

where J_n has the form

$$J_n = \sum a_{r,s,\vec{i}}\,\epsilon^r\chi_{i_1}\cdots\chi_{i_{2r}}\chi'_{i_{2r+1}}\cdots\chi'_{i_{2(s+r)+1}}$$

$$\times \delta_{i_1+\cdots+i_{2(s+r)+1},\,n}. \qquad (2.32)$$

Here, the expansion coefficients, $a_{r,s,\vec{i}}$, are numbers of order one completely determined by expanding the square-root in Eq. (2.16), and $r+s \geq 1$. Note that J_n is determined from lower order solutions of χ in perturbation theory. This allows us to treat J_n as a known source in the nth order equation of motion for χ. Thus we can solve Eq. (2.31) iteratively for the χ_n.

Our central claim is that to any order in perturbation theory χ has the form

$$\chi_n = \sum b_{n,r,s,t}^{\vec{}}\,\lambda^{n_1}(\lambda y)^{n_2}(\lambda y_c)^{n_3}(\lambda/\Delta_-)^{n_4}e^{r\Delta_+(y-y_c)}$$

$$\times e^{s\Delta_- y}e^{t\Delta_- y_c}, \qquad (2.33)$$

where the constant coefficients b are order one or smaller (they can contain positive but not negative powers of ϵ), and n_i, r, s, t are integers such that

$$n_1 \geq 1, \quad r, n_2, n_3, n_4 \geq 0$$

$$n = n_1 + n_2 + n_3 + n_4$$

$$|s|, |t| \leq n. \qquad (2.34)$$

Let us discuss the significance of this claim. First, it shows that the perturbative expansion is well-behaved. Note that this is *a priori* not guaranteed. For example, the formal small parameter λ could be overwhelmed by a factor of $e^{1/\epsilon}$ or a high enough power of $1/\epsilon$. However, it is straightforward to see that this is not so given Eq. (2.33). Although $y \lesssim y_c \sim \mathcal{O}(1/\Delta_-) \sim \mathcal{O}(1/\epsilon)$, powers of y, y_c, and $1/\Delta_-$ are accompanied by powers of the parametrically smaller λ. While positive powers of $e^{\Delta_+ y} > 1$ appear, they are compensated by powers of $e^{-\Delta_+ y_c}$. While arbitrary powers of $e^{\Delta_- y}$ can appear, since we will show Eq. (2.17) self-consistently, $e^{\Delta_- y} \sim \mathcal{O}(1)$. Secondly, Eq. (2.17) also follows since it is clear from Eq. (2.33) that χ is dominated by χ_1 for all y. Therefore, the second order determination we made for A' dominates, and leads to a stable value of y_c near that of Eq. (2.27). The hidden brane junction condition also receives subleading corrections which can be satisfied by fine-tuning ρ_h order by order in λ. This is the higher-orders incarnation of the four-dimensional cosmological constant fine-tuning problem of ensuring a 4D Poincaré invariant vacuum solution.

We will now prove our claim (2.33) for χ_n by induction in n. First we note that the claim is true for $n=1$ based on our explicit solution. For a general perturbative order, n, we assume that the claim is true for all lower orders. Then clearly J_n, constructed from the lower order solution for χ, also has the form

$$J_n = \sum c_{n,r,s,t}^{\vec{}}\,\lambda^{n_1}(\lambda y)^{n_2}(\lambda y_c)^{n_3}(\lambda/\Delta_-)^{n_4}e^{r\Delta_+(y-y_c)}$$

$$\times e^{s\Delta_- y}e^{t\Delta_- y_c}, \qquad (2.35)$$

where the constant coefficients, c, are also order one or smaller. Given this source term we can solve Eq. (2.31),

$$\chi_n = \int_0^{y_c} G(y,y')J_n(y')dy', \qquad (2.36)$$

where G is the Green function satisfying

$$\left\{\frac{d^2}{dy^2} - 4\frac{d}{dy} - \epsilon - \mu_v\delta(y-y_c) - \mu_h\delta(y)\right\}G(y,y')$$

$$= \delta(y-y'), \qquad (2.37)$$

subject to orbifold boundary conditions.

The detailed form of the Green function is straightforwardly worked out. However, in order to complete our induction we do not require the full details, but only the general form,

$$G(y,y') = \begin{cases} (\mathcal{O}(1)e^{-\Delta_+ y'} + \mathcal{O}(1)e^{\Delta_-(y_c-y')-\Delta_+ y_c})e^{\Delta_+ y} + (\mathcal{O}(1)e^{-\Delta_+ y'} + \mathcal{O}(1)e^{\Delta_-(y_c-y')-\Delta_+ y_c})e^{\Delta_- y}, & 0 < y < y', \\ (\mathcal{O}(1)e^{\Delta_-(y_c-y')-\Delta_+ y_c} + \mathcal{O}(1)e^{\Delta_- y_c-\Delta_+(y'+y_c)})e^{\Delta_+ y} + (\mathcal{O}(1)e^{-\Delta_- y'} + \mathcal{O}(1)e^{-\Delta_+ y'})e^{\Delta_- y}, & y' < y < y_c. \end{cases}$$

$$(2.38)$$

The proof that χ_n satisfies the claim of Eq. (2.33) then follows by inspection of all the possible terms that can arise from Eq. (2.36) given Eqs. (2.35) and (2.38). The basic integrals involved in evaluating Eq. (2.36) are of the form

$$\int_{y_1}^{y_2} dy' \, y'^m = \frac{y_2^{m+1} - y_1^{m+1}}{m+1}, \qquad (2.39)$$

or

$$\int_{y_1}^{y_2} dy' \, y'^m e^{\Delta y'} = \frac{d^m}{d\Delta^m}\left(\frac{e^{\Delta y_2} - e^{\Delta y_1}}{\Delta}\right), \qquad (2.40)$$

where m is a non-negative integer, Δ is some linear combination of $\Delta_{\pm}$ with integer coefficients, and $y_1 = 0, y; \; y_2 = y, y_c$. It is straightforward but a little tedious to then check that all possible terms arising in Eq. (2.36) do indeed satisfy the claim of Eq. (2.33).

3. The case of $\lambda > \epsilon$

We conclude this reworking of the Goldberger-Wise mechanism by discussing the relationship between λ and ϵ. Thus far, we have considered λ, our formal perturbative expansion parameter, to be formally smaller than ϵ. The reason for doing this is because of terms in Eq. (2.33) with non-zero n_2, n_3 or n_4, all of which would signify effects of order $(\lambda/\epsilon)^{n_i}$. The presence of such terms threatens to invalidate perturbation theory if we allowed $\lambda > \epsilon$. However, we can in fact prove a stronger result, that all such terms in Eq. (2.33), are in fact accompanied by a suppression factor of order $\epsilon^{n_2+n_3+n_4}$, so that we can take take $\lambda > \epsilon$ (though formally small compared to unity). This relationship is indeed implicit in the original analysis of Goldberger and Wise. We have not emphasized this fact until now both for simplicity and because the suppression factor is not generally true when we study higher derivative perturbations. We will study the special circumstances under which higher-derivative perturbations do not require λ/ϵ to be small in Sec. V.

The stronger result, that there are accompanying factors of order $\epsilon^{n_2+n_3+n_4}$, is again proven by induction on n, noting first that it trivially holds for $n = 1$. For a larger n, we assume it holds for lower orders. Therefore, J_n shares the same property since it is made from products of (lower-order) χ and χ'. But χ_n determined by Eq. (2.36) can generate (at most) one new power of y, y_c or $1/\Delta_-$ relative to J_n: a new power of y or y_c can arise as in Eq. (2.39) or Eq. (2.40), or one new power of $1/\Delta_-$ can arise as in Eq. (2.40) in case $\Delta = \Delta_-$. One can check (somewhat tediously) that such cases can only arise from terms in J_n involving at least one power of χ

without a derivative acting on it [$r > 1$ in Eq. (2.32)]. Otherwise, in terms where the integrals (2.39), (2.40) produce an extra power of y, y_c or $1/\Delta_-$, the derivatives in χ' always bring down a power of $\Delta_- \sim \mathcal{O}(\epsilon)$ or eliminate a power of y' so that in fact χ_n does satisfy the claim of the induction. But for terms in J_n with at least one non-derivative power of χ, we see in Eq. (2.32) that there are explicit powers of ϵ arising in the expansion (2.32). These compensate the single power of of $\mathcal{O}(1/\epsilon)$ that can be generated in χ_n, so the induction claim still goes through.

III. SYMMETRIES OF INTERACTIONS

Our task now is to specify which higher-derivative interactions are allowed in generalizing RS1 and the Goldberger-Wise mechanism, in particular how they are constrained by symmetries. Since the spacetime manifold is topologically $\mathbf{R}^4 \times S^1/\mathbf{Z}_2$, the relevant symmetries are 5D general coordinate invariance and the $\mathbf{Z}_2$ parity symmetry. The coordinate-invariant or geometric statement of the $\mathbf{Z}_2$ symmetry is as follows. We first start with the manifold $\mathbf{R}^4 \times S^1$ and consider two "3-branes" which divide the S^1 into two disjoint regions. We choose geometries and scalar fields on $\mathbf{R}^4 \times S^1$ such that the two regions are reflection symmetric. We now wish to find a convenient description of all interactions which respect this.

It is useful to start with a formalism which respects full 5D general coordinate invariance on $\mathbf{R}^4 \times S^1$, even though this necessarily involves coordinate systems which do not respect the $\mathbf{Z}_2$-symmetry of the invariant geometry. (However, see Ref. [12] for an alternative symmetry implementation.) The two fixed-point branes ("hidden" and "visible") will be coordinatized as $Y_{hid}^M(x), Y_{vis}^M(x)$, where x are parameterizations of the 3-branes. Using these brane fields, a 5D metric, $G_{MN}(X)$, and 5D Goldberger-Wise scalar $\chi(X)$, we can form fully 5D general coordinate-invariant bulk and brane actions on $\mathbf{R}^4 \times S^1$, as in Ref. [13].

Once a generally coordinate invariant action has been chosen in this way, we can "gauge-fix" by choosing our 5D coordinates, X^M, to consist of x^μ and an extra-dimensional angle, $-\pi \le \phi \le \pi$, such that

$$Y_{hid}^\mu(x) \equiv Y_{vis}^\mu(x) \equiv x^\mu$$

$$Y_{hid}^\phi \equiv 0$$

$$Y_{vis}^\phi \equiv \pi, \qquad (3.1)$$

and where the symmetry of the geometry and χ is manifest,

$$G_{\mu\nu}(x,\phi)=G_{\mu\nu}(x,-\phi)$$

$$G_{\mu\phi}(x,\phi)=-G_{\mu\phi}(x,-\phi)$$

$$G_{\phi\phi}(x,\phi)=G_{\phi\phi}(x,-\phi)$$

$$\chi(x,\phi)=\chi(x,-\phi). \tag{3.2}$$

Our procedure for writing actions on $\mathbf{R}^4 \times S^1/\mathbf{Z}_2$ is therefore to (a) write a general 5D coordinate invariant action for bulk and brane fields on $\mathbf{R}^4 \times S^1$ as detailed in Ref. [13], (b) gauge-fix the action according to Eqs. (3.1) and (3.2), using the $\mathbf{Z}_2$-symmetry of the allowed configurations. Once this is done the brane-fields $Y_{hid}^M(x), Y_{vis}^M(x)$, no longer explicitly appear. Finally, (c) one must check that all terms in the action are compatible with orbifolding, namely they are invariant under

$$G_{\mu\nu}(x,\phi)\rightarrow G_{\mu\nu}(x,-\phi)$$

$$G_{\mu\phi}(x,\phi)\rightarrow -G_{\mu\phi}(x,-\phi)$$

$$G_{\phi\phi}(x,\phi)\rightarrow G_{\phi\phi}(x,-\phi)$$

$$\chi(x,\phi)\rightarrow \chi(x,-\phi), \tag{3.3}$$

for general metrics and scalar field.

It is straightforward to check that it does not matter whether one imposes the $\mathbf{Z}_2$-symmetry and gauge-fixing on the action or varies the un-gauge-fixed action without imposing the $\mathbf{Z}_2$-symmetry, as long as one then imposes the $\mathbf{Z}_2$ and gauge-fixing on the equations of motion. For convenience we will assume that the $\mathbf{Z}_2$-symmetry and gauge-fixing have been imposed at the level of the action, as implicit in the RS1 and Goldberger-Wise papers.

Recall that the central reason we wish to orbifold is that we wish to take the visible brane to have "negative tension," which would normally yield a ghost-like sign for the associated Y_{vis}^ϕ kinetic term (Ref. [13]), causing a vacuum instability towards violent brane fluctuations. However, such brane fluctuations violate the $\mathbf{Z}_2$-symmetry, and therefore are eliminated by orbifolding. It may seem that by formally reintroducing Y_{vis}^ϕ in step (a) of our procedure for generating allowed action terms, we are reintroducing the instability. However this is not so. The $\mathbf{Z}_2$-symmetry renders Y_{vis}^ϕ as pure gauge, with no physical import, as reflected in the gauge fixing of step (b).

IV. HIGHER-DERIVATIVE OPERATORS IN EFFECTIVE BRANE THEORIES

In any effective field theory we expect there to be higher-derivative operators which are remnants of the more fundamental physics which has been integrated out. Their interpretation and treatment is generally well-understood. However, they pose special problems when they appear in effective theories involving orbifolds and branes in extra dimensions. These stem from their appearance as δ-function sources in the extra dimension, rather than some structure with a finite size (perhaps due to quantum-mechanical or stringy effects).

High derivatives applied to such δ-functions produce messy and ill-defined equations of motion. We will show how such problems can be treated by a combination of field redefinitions and classical brane renormalization. See Ref. [14] for a recent discussion of brane renormalization focusing on co-dimension 2, and Refs. [15] for earlier work on classical renormalization.

A. The formal derivative expansion

To keep only the essential considerations in focus we will not consider gravity here. We will limit ourselves to a single scalar field, χ, living on an extra-dimensional $S^1/\mathbf{Z}_2$ of fixed radius r_c (and the usual four dimensions). The same considerations apply in the presence of gravity or non-trivial warp factor. As before we take χ to be dimensionless and even under the orbifold parity. The general Lagrangian then has the form

$$\mathcal{L}=M^3\left\{\frac{K(\chi)}{2}(\partial_M\chi)^2-V(\chi)-v_i(\chi)\delta(y-y_i)\right.$$

$$\left.+\sum_{n>0}\frac{1}{M^n}\mathcal{L}_{h.d.}^{(n)}(\chi,\partial_M)\right\}, \tag{4.1}$$

where M is the only explicit scale appearing in order to balance dimensions, and y is now defined as the extra-dimensional coordinate corresponding to proper distance in the extra dimension. Brane localized terms are represented by multiplying by one of $\delta(y-y_i)$, where $y_1=0, y_2=\pi r_c$. $\mathcal{L}_{h.d.}$ are arbitrary higher-derivative terms organized in powers of $1/M$. Note that derivatives always appear in even numbers due to the orbifold parity and 4D Lorentz invariance, so that the $1/M$-suppressed terms (once the overall M^3 is excluded) are those with more than two derivatives in the bulk and those with any derivatives on a brane. We have not separated out brane and bulk higher-derivative terms although we will do this later.

We will study the equations of motion subject to the 4D Poincaré ansatz, $\chi=\chi(y)$,

$$-\partial_y(K(\chi)\partial_y\chi)-\frac{K'(\chi)}{2}(\partial_y\chi)^2+V'(\chi)+v_i'(\chi)\delta(y-y_i)$$

$$=\sum_{n>0}\frac{1}{M^n}\frac{\delta S_{h.d.}^{(n)}}{\delta\chi}. \tag{4.2}$$

This would appear to be a differential equation of arbitrarily high order and therefore requiring an arbitrary number of initial conditions to solve. However, there is a unique solution which is perturbatively close, in a $(\partial/M)^m$ expansion, to the zeroth order solution,

$$-\partial_y(K(\chi_0)\partial_y\chi_0)-\frac{K'(\chi_0)}{2}(\partial_y\chi_0)^2+V'(\chi_0)$$

$$+v_i'(\chi_0)\delta(y-y_i)=0. \tag{4.3}$$

This zeroth order equation is manifestly a second order differential equation subject to orbifold boundary conditions, which we know has a distinct solution. Of course, in order to determine χ_0 we may have to approximate as we did for the Goldberger-Wise mechanism, perturbing in λ, but in order to focus on higher derivative perturbations let us take the zeroth order solution, χ_0, as given.

An expansion in ∂_y/M may seem sensible in the bulk where solutions are smooth but ill-defined in the presence of the brane δ-functions. We will show how these ill-defined brane singularities can be renormalized away. To get started and to understand the issues we will proceed bravely and formally define the ∂_y/M perturbation expansion.[2] We will work inductively in n. Let the expansion of χ in powers of $1/M$ be written

$$\chi = \sum_m \chi_m. \qquad (4.4)$$

Suppose that we can solve Eq. (4.2) to order $1/M^m$ (we know we can do this for $m=0$). We will then show how to construct a solution up to order $1/M^{m+1}$.

We substitute Eq. (4.4) into Eq. (4.2) and focus on the term precisely of order $1/M^{m+1}$,

$$\left\{ -\partial_y K(\chi_0)\partial_y - \partial_y(\partial_y\chi_0)K'(\chi_0) + \frac{1}{2}K''(\chi_0)(\partial_y\chi_0)^2 \right.$$
$$+ K'(\chi_0)(\partial_y\chi_0)\partial_y + V''(\chi_0) + v_i''(\chi_0)\delta(y-y_i) \Big\} \chi_{m+1}$$
$$= \frac{\delta S}{\delta\chi}\left[\sum_{l\leq m}\chi_l(y) \right]\Big|_{m+1}, \qquad (4.5)$$

where the right-hand side is the functional derivative of the *entire* action, including the higher derivative terms, evaluated for the field $\Sigma_{l\leq m}\chi_l(y)$, but keeping precisely the terms of order $1/M^{m+1}$. Thus, in order to perturbatively improve our solution by one order in $1/M$, that is solve for χ_{m+1} in terms of $\chi_{l\leq m}$, we only need to solve the above linear second order equation for χ_{m+1} subject to orbifold boundary conditions, with a source term determined by the lower order solution $\chi_{l\leq m}$. This we can do

$$\chi_{m+1}(y) = \int dy' G(y,y') \frac{\delta S}{\delta\chi}\left[\sum_{l\leq m}\chi_l(y') \right]\Big|_{m+1}, \qquad (4.6)$$

where G is the Green function determined by orbifold boundary conditions satisfying

$$\left\{ -\partial_y K(\chi_0)\partial_y - \partial_y(\partial_y\chi_0)K'(\chi_0) + \frac{1}{2}K''(\chi_0)(\partial_y\chi_0)^2 \right.$$
$$+ K'(\chi_0)(\partial_y\chi_0)\partial_y + V''(\chi_0)$$
$$+ v_i''(\chi_0)\delta(y-y_i) \Big\} G(y,y') = \delta(y-y'). \qquad (4.7)$$

B. The problem

Such Green functions, G (and also χ_0) will clearly be smooth in the bulk, with absolute-value type kinks at the branes. That is, their first derivatives will have step-function discontinuities on the branes and their second derivatives will have δ-functions on the branes. Let us suppose this property of the Green function is shared by $\chi_{l\leq m}(y)$ and see how trouble can arise in χ_{m+1}. First, consider bulk terms that can appear in $(\delta S/\delta\chi)[\Sigma_{l\leq m}\chi_l(y')]|_{m+1}$. If there are more than two derivatives acting on the same field, then we will have derivatives of δ-functions on the branes which we must convolute with G, which has absolute-value type kinks. The result is therefore ill defined. Similarly, if we have a power greater than one of second derivatives of fields we will have to integrate products of δ-functions which is again ill defined. On the other hand, if there is at most a single field with two derivatives acting on it, multiplied by any number of fields with at most first derivatives, then we will only have to integrate a single δ-function multiplied by a function with absolute-value type kinks.[3] This does yield a well-defined χ_{m+1}, which is also smooth except for absolute-value type kinks on the branes. It is rather straightforward to see that the troublesome higher-derivative bulk terms in the equations of motion correspond precisely to bulk higher-derivative terms in the Lagrangian, Eq. (4.1), with more than one derivative acting on a field (unless it can be eliminated by integration by parts).

Let us now turn to brane terms on the right-hand side of Eq. (4.6), again assuming that $\chi_{l\leq m}(y)$ has only absolute-value type kinks and considering the effects on $\chi_{m+1}(y)$. It is clear that if in the Lagrangian, Eq. (4.1), there are any derivatives in brane terms, then in $(\delta S/\delta\chi)[\Sigma_{l\leq m}\chi_l(y')]|_{m+1}$ we will have derivatives of δ functions which makes Eq. (4.6) ill-defined again. On the other hand if there are no derivatives on the brane terms, they will give simple δ-functions to be integrated in Eq. (4.6), resulting in a χ_{m+1} with only absolute-value type kinks.

Below, we will show that an arbitrary higher-derivative Lagrangian, Eq. (4.1), can be massaged so that all bulk terms have at most one ∂_y acting on any given field, e.g. $\chi^8(\partial\chi)^{10}$, and all brane terms have no ∂_y's at all. As shown above, this will result in a well-defined perturbative derivative expansion for solving the equations of motion, with solutions having at most absolute-value type kinks. The reader may on a

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[2]This is in the same spirit as setting up the formal Feynman diagram series in quantum field theory, which is also ill-defined until after regularization and renormalization.

[3]The first derivatives of fields actually have step-function discontinuities, but because of the orbifold symmetries they must always appear in even powers, yielding a function with only absolute-value type kinks on the branes.

first reading wish to accept that this can be done, skip the rest of this section, and continue with the stability analysis for higher-derivative bulk terms subject to the above conditions.

C. Bulk higher-derivative operators

In this section we will show how higher-derivative operators in the bulk can be massaged using field redefinitions. To simplify the considerations note that the equations of motion subject to the 4D Poincaré ansatz, $\chi = \chi(y)$, can always be obtained by first imposing the ansatz on the action, Eq. (4.1), and then functionally differentiating with respect to $\chi(y)$. Doing this we can write the bulk part of the Lagrangian in the form

$$\mathcal{L}_{bulk} = M^3 \left\{ -\frac{1}{2} K(\chi)(\partial_y \chi)^2 - V(\chi) \right.$$
$$\left. + \sum_{\vec{n}} \frac{a_{\vec{n}}(\chi)}{M^{|n|}} (\partial \chi)^{n_1} \dots (\partial^N \chi)^{n_N} \right\}, \qquad (4.8)$$

where $|n| \equiv n_1 + 2n_2 \cdots + N n_N - 2 > 0$, and where now $\partial \equiv \partial_y$.

As already discussed, the problematic terms are those with $N > 1$. To begin, we assign the bulk potential a formal strength $V \sim \mathcal{O}(k^2) < M^2$ and augment our effective field theory derivative expansion with an expansion in k/M. We then make a field redefinition, $\psi(\phi)$, to render the kinetic term "canonical,"

$$\psi \equiv \int d\chi K^{1/2}(\chi). \qquad (4.9)$$

Then we have a form

$$\mathcal{L}_{bulk} = r_c M^3 \left\{ \frac{1}{2} (\partial \psi)^2 - V(\psi) \right.$$
$$\left. + \sum_{\vec{n}} \frac{a_{\vec{n}}(\psi)}{M^{|n|}} (\partial \psi)^{n_1} \dots (\partial^N \psi)^{n_N} \right\}, \qquad (4.10)$$

where of course V and a_n are redefined too. This particular field redefinition will simplify stating our procedure, and will be undone at the end.

We begin by working to zeroth order in k/M (that is, we can neglect V), but to some non-zero order, m, in ∂/M. Working inductively, we assume that by some field redefinitions of ψ all $N > 1$ terms have been eliminated from all terms of lower than m-th order. We then make the following field redefinition:

$$\psi \to \psi + \sum_{|n|=m} (-\partial)^{N-2} \frac{a_{\vec{n}}(\psi)}{M^{|n|}} (\partial \psi)^{n_1} \dots (\partial^N \psi)^{n_N - 1}. \qquad (4.11)$$

Note that this transformation is only sensible for $N > 1$. To m-th order, the only difference this makes to the Lagrangian arises from substituting into the zeroth order

kinetic term, whereupon (after integrating by parts) it produces precisely the term needed to cancel $\Sigma_{|n|=m}[a_{\vec{n}}/(\psi)M^{|n|}](\partial \psi)^{n_1} \dots (\partial^N \psi)^{n_N}$ in the Lagrangian, plus terms of higher than m-th order. Thus one uses this procedure to eliminate $N > 1$ terms to any desired order in ∂/M, when we neglect $\mathcal{O}(k^2)$.

More generally, the field transformation above will, however, reintroduce terms of various orders in ∂/M, but now with coefficients of order k^2/M^2. But we can once again do a field redefinition at order k^2/M^2 and work to any desired order in ∂/M to eliminate $N > 1$ terms. This will in turn induce terms of order k^4/M^4. In this way, one can eliminate all $N > 1$ terms to any fixed order in k^2/M^2 and ∂/M. Finally, we can transform back to a field, χ_{new},

$$\frac{\partial \psi_{new}}{\partial \chi_{new}} = K^{1/2}(\chi_{new}), \qquad (4.12)$$

in terms of which

$$\mathcal{L}_{bulk} = r_c M^3 \left\{ \frac{1}{2} K(\chi_{new})(\partial \chi_{new})^2 - V(\chi_{new}) \right.$$
$$\left. + \sum_{n} \frac{a_n(\chi_{new})}{M^{n-2}} (\partial \chi_{new})^n \right\}. \qquad (4.13)$$

We have thereby removed all the problematic bulk terms discussed in the previous subsection.

It can be shown (again, tediously) that our field redefinition in Poincaré ansatz,

$$\chi(y) \to \chi_{new}(y), \qquad (4.14)$$

can be lifted to a fully 5D covariant transformation,

$$\chi(x,y) \to \chi_{new}(x,y), \qquad (4.15)$$

which, however, reduces to $\chi(y) \to \chi_{new}(y)$ upon imposing the 4D Poincaré ansatz. Similarly, when gravity is included there are fully 5D generally covariant field redefinitions which upon imposing the Poincaré ansatz ensure that all $N > 1$ terms (terms with more than one ∂_y acting on a field) are absent.

D. Brane higher-derivative operators

It is obvious that the field transformations which we discussed above to massage the bulk action will induce derivative terms on the orbifold fixed points. Furthermore, such derivative terms may already be present anyway. Therefore after ensuring that all bulk terms satisfy $N \leq 1$, we must still massage away brane terms in the Lagrangian of the form

$$\mathcal{L}_{brane} = \int dy \, \delta(y - y_i) M^3 k$$
$$\times \left\{ \sum_{\vec{n}:N>0} \frac{a_{\vec{n}}(\chi)}{M^{|n|}} (\partial \chi)^{n_1} \dots (\partial^N \chi)^{n_N} \right\}. \qquad (4.16)$$

We will show that the important physics contained in χ away from the branes can be obtained by using effective renormalized brane Lagrangians without any extra-dimensional derivatives,

$$\mathcal{L}_{eff.\ brane} = \int dy\, \delta(y - y_i) M^3 k v_i(\chi). \qquad (4.17)$$

The first step before renormalizing the original Lagrangian is to regulate it, say by the replacement,

$$\delta(y - y_i) \rightarrow D(y - y_i)$$
$$\equiv N e^{-1/(y - y_i + t)} e^{1/(y - y_i - t)}, \quad -t < y < t$$
$$0 \ \text{else}, \qquad (4.18)$$

where

$$N^{-1} = \int_{-t}^{t} dy\, e^{-1/(y + t)} e^{1/(y - t)}. \qquad (4.19)$$

This replaces the infinitesimally thin brane by a thick brane of thickness $t : 1/M < t \ll r_c$. Note that the brane profile $D(y - y_i)$ is a smooth function (that is, a C^∞ function) of compact support.

With this regulator in place, clearly our perturbative derivative expansion is always well-defined despite derivatives on branes, and Eq. (4.6) will always give smooth solutions. Of course, the results depend on our choice of regulator. However we will show that away from the (compact) core of the branes, this regulator-dependence can be completely subsumed into an effective brane Lagrangian, Eq. (4.17).

For simplicity of exposition we will consider a single orbifold fixed point on $\mathbf{R}/\mathbf{Z}_2$ rather than the two fixed points of $S^1/\mathbf{Z}_2$. The generalization to $S^1/\mathbf{Z}_2$ is entirely straightforward. First, recall that we have shown that at every order in perturbation theory we only solve second order differential equations. Thus the solution $\chi(y)$ in perturbation theory is completely specified by $\chi(0)$ and $\partial_y \chi(0)$. Orbifold parity sets $\partial_y \chi(0) = 0$, but $\chi(0)$ (on $\mathbf{R}/\mathbf{Z}_2$ not $S^1/\mathbf{Z}_2$) is an unfixed number, $\chi(0) = c$. In particular by solving the equations of motion, $\chi(t) = \chi(-t)$ and $\partial_y \chi(t) = -\partial_y \chi(-t)$ are both functions of c,

$$\chi(t) = f(c)$$
$$\partial_y \chi(t) = g(c). \qquad (4.20)$$

Let us define $\widetilde{\chi}(y)$, to be the result of integrating the second order differential equations from t to a general $y > 0$, using $\chi(t), \partial_y \chi(t)$ as boundary conditions, but completely neglecting brane interactions. Now for $y > t$, $\widetilde{\chi}(y)$ is in fact the correct solution $\chi(y)$, since the neglected brane interactions vanish in this region. But for $y < t$, clearly $\widetilde{\chi}(y)$ cannot be trusted. Nevertheless, as long as the physics of interest is dominated by bulk field behavior outside the core of the brane, we can use $\widetilde{\chi}(y)$. This will be the case for the solution to the hierarchy problem, which is determined by the warp factor accumulated over the large bulk $(r_c \gg 1/M)$.

Let us define

$$\widetilde{\chi}(0_+) = F(\chi(t), \partial_y \chi(t))$$
$$\partial_y \widetilde{\chi}(0_+) = G(\chi(t), \partial_y \chi(t)). \qquad (4.21)$$

We can eliminate dependence on $\chi(t), \partial_y \chi(t)$ in favor of c,

$$\widetilde{\chi}(0_+) = F(f(c), g(c)) \equiv p(c)$$
$$\partial_y \widetilde{\chi}(0_+) = G(f(c), g(c)) \equiv q(c). \qquad (4.22)$$

Therefore, order by order in perturbation theory we can eliminate dependence on c by inverting p,

$$\partial_y \widetilde{\chi}(0_+) = q(p^{-1}[\widetilde{\chi}(0_+)]). \qquad (4.23)$$

We will now show that $\widetilde{\chi}$ is the classical solution corresponding to a "renormalized" effective Lagrangian with the same bulk terms as before, but with a δ-function brane-localized potential term (that is without any extra-dimensional derivatives). Recall that by field redefinitions we have already ensured that the bulk Lagrangian only depends on χ and its first derivative, $\mathcal{L}_{bulk}(\chi, \chi')$. The full effective Lagrangian, including the effective brane potential, is then given by

$$\mathcal{L}_{eff} = \mathcal{L}_{bulk}(\chi, \chi') + \delta(y) v_{eff}(\chi), \qquad (4.24)$$

where

$$v_{eff}(\chi) = \int d\chi\ 2q(p^{-1}(\chi)) \frac{\partial^2 \mathcal{L}_{bulk}}{(\partial \chi')^2}[\chi, q(p^{-1}(\chi))]. \qquad (4.25)$$

The reader can straightforwardly check that the equation of motion that follows from this effective Lagrangian is equivalent to the equation of motion due only to the bulk terms away from $y = 0$, supplemented by the boundary condition, Eq. (4.23).

Thus, we are always able to find an effective Lagrangian of the form (4.24), which has solutions which agree with those of a general Lagrangian supplemented by a regulator, outside the regulated core of the branes. The important physics such as the hierarchy are insensitive to the general disagreement inside the thickness of the regulated brane. Note that the effective Lagrangian does not suffer from UV ambiguities, and therefore needs no regulator. In that sense it is "renormalized."

V. STABILITY AND HIGHER DERIVATIVE OPERATORS

A. Setup of the model

We will now show that the Goldberger-Wise mechanism can be realized in a very general setting, including higher-derivative operators. We will take the higher-derivative terms to be suppressed by appropriate powers of $1/M$ and constrained by symmetries according to the discussion of Sec. III. Furthermore, we will assume that they have already been massaged by field redefinitions and brane-renormalization as

discussed in the previous section, so that fields appearing in bulk interactions have at most one extra-dimensional derivative acting on them, while there are no extra-dimensional derivatives in brane terms.

The action in Einstein frame is

$$S_{bulk} = M^3 \int d^4x \int_{-\pi}^{\pi} d\phi \sqrt{G} \left(k^2 V(\chi) - \frac{1}{4} R + \frac{1}{2} K(\chi)(D\chi)^2 \right.$$
$$\left. + M^2 \mathcal{L}_{h.d.} \left(G_{MN}, \chi, \frac{\partial_N}{M} \right) \right) \tag{5.1}$$

$$S_{vis} = -M^3 k \int d^4x \sqrt{g_{vis}} v_v(\chi) \tag{5.2}$$

$$S_{hid} = -M^3 k \int d^4x \sqrt{g_{hid}} v_h(\chi). \tag{5.3}$$

Here, $\mathcal{L}_{h.d.}$ indicates terms containing more than two derivatives with dimensionless coefficients of order unity and suppressed by powers of $1/M$. We can expand the dimensionless functions V, K, v_v, and v_h in powers of χ

$$V(\chi) = 3 - \frac{1}{2} \epsilon \chi^2 + \mathcal{O}(\chi^3)$$

$$K(\chi) = 1 + \mathcal{O}(\chi)$$

$$v_v(\chi) = \rho_v + \lambda_v \chi + \frac{1}{2} \mu_v \chi^2 + \mathcal{O}(\chi^3)$$

$$v_h(\chi) = \rho_h + \lambda_h \chi + \frac{1}{2} \mu_h \chi^2 + \mathcal{O}(\chi^3). \tag{5.4}$$

We will perform perturbation theory in powers of $k^2/M^2 \ll 1$ and the brane-tadpoles, $\lambda_{v,h}$, as discussed in Sec. II. Higher-derivative terms are automatically suppressed by powers of k^2/M^2 because the bulk potential is dominated by a 5D cosmological constant of order $M^3 k^2$. Formally, this is seen by the fact that in our dimensionless y-coordinate every ∂_y is accompanied by k. For simplicity, we take k^2/M^2 and $\lambda_{v,h}$ to all be of the order of our formal small parameter, λ. We continue to take the relations of Eq. (2.6) to hold among $\lambda_{v,h}, \rho_{v,h}, \mu_{v,h}$ and ϵ.

We now write the equations of motion with the Poincaré ansatz using again the dimensionless variable $y = kr_c \phi$ as in Sec. II,

$$\chi'' + 4A'\chi' - \epsilon\chi = (\lambda_v + \mu_v\chi + \mathcal{O}(\chi^2))\delta(y - y_c)$$
$$+ (\lambda_h + \mu_h\chi + \mathcal{O}(\chi^2))\delta(y)$$
$$+ f_1(\chi, \chi', A') + \chi'' f_2(\chi, \chi', A') \tag{5.5}$$

$$A'' + \frac{2}{3}(\chi')^2 = -\frac{2}{3} \left(\rho_v + \lambda_v\chi + \frac{1}{2} \mu_v\chi^2 + \mathcal{O}(\chi^3) \right)$$
$$\times \delta(y - y_c) - \frac{2}{3} \left(\rho_h + \lambda_h\chi + \frac{1}{2} \mu_h\chi^2 \right.$$
$$\left. + \mathcal{O}(\chi^3) \right) \delta(y) + f_3(\chi, \chi', A')$$
$$+ A'' f_4(\chi, \chi', A') \tag{5.6}$$

$$(A')^2 = 1 - \frac{1}{6} \epsilon\chi^2 + \frac{1}{6}(\chi')^2 + f_5(\chi, \chi', A'), \tag{5.7}$$

where the f_i refer to the variation of the bulk higher-derivative action subject to the special form discussed in Sec. IV. Note that the right-hand side of Eq. (5.7) does not contain terms with delta functions or second derivatives.

Perturbation theory follows along lines similar to Sec. II. We first formally expand A and χ as in Eq. (2.14) and substitute into the equations of motion. We then solve Eq. (5.7) for the A'_n in terms of the $\chi_{m<n}$ and $\chi'_{m<n}$ to any desired order. Substituting for the A'_n into Eq. (5.5) then yields an equation involving only the χ_n and χ'_n which we solve perturbatively. Then, by 5D general covariance in the bulk, Eq. (5.6) is automatically solved, except for the two brane junction conditions which are solved by fine-tuning ρ_h and setting y_c to its stable vacuum value.

To see that Eq. (5.6) is automatically satisfied up to junction conditions, note that under infinitesimal general coordinate transformations the bulk action is invariant,

$$S_{bulk}[G_{MN} + D_M\eta_N + D_N\eta_M, \chi + \partial^M\chi\eta_M] = S_{bulk}[G_{MN}, \chi], \tag{5.8}$$

for arbitrary infinitesimal η_M. This implies

$$2D_M \left(\frac{1}{\sqrt{G}} \frac{\delta S_{bulk}}{\delta G_{MN}} \right) - \frac{1}{\sqrt{G}} \frac{\delta S_{bulk}}{\delta\chi} \partial^N\chi = 0. \tag{5.9}$$

In the Poincaré ansatz this gives

$$\partial_5 \left(\frac{\delta S_{bulk}}{\delta G_{55}} \right) = A' \frac{\delta S_{bulk}}{\delta G_\mu^\mu} - \frac{1}{2} \chi' \frac{\delta S_{bulk}}{\delta\chi}. \tag{5.10}$$

This shows that Eq. (5.6), the $G_{\mu\nu}$ equation of motion, is satisfied up to junction conditions, given Eq. (5.5), the χ equation of motion, and Eq. (5.7), the G_{55} equation of motion.

The equation for χ obtained by eliminating A' from Eq. (5.5) using Eq. (5.7) has the form (2.31), but now with sources following from Eqs. (5.5)–(5.7) of the form:

$$J_n = J_n^{bulk} + J_n^{brane} + J_n^{mixed}, \tag{5.11}$$

where

$$J_n^{bulk} = \sum a_{r,s,\vec{i}} \lambda^m \chi_{i_1} \cdots \chi_{i_r} \chi'_{i_{r+1}} \cdots \chi'_{i_s} \delta_{i_1 + \cdots + i_s + m, \ n}$$
(5.12)

$$J_n^{brane_j} = \sum b_{r,s,\vec{i}} \lambda^m \chi_{i_1} \cdots \chi_{i_s} \delta_{i_1 + \cdots i_s + m, n} \delta(y - y_j),$$
(5.13)

and

$$J_n^{mixed} = \sum c_{r,s,\vec{i}} \lambda^m \chi_{i_1} \cdots \chi_{i_s} \chi'_{i_{r+1}} \cdots \chi'_{i_s} \chi''_p$$

$$\times \delta_{i_1 + \cdots i_s + m + p, \ n}.$$
(5.14)

This last term acts as a brane term as well as a bulk term because χ'' contains δ-functions at the branes as well as a smooth bulk behavior. Again, the constant coefficients, a, b and c are order one or smaller. The nth order solution is obtained using Eq. (2.36) with the same Green function defined by Eq. (2.37).

B. Stabilization at second order

We will show here that hierarchy stabilization satisfying Eq. (2.17) is naturally achieved by second order in λ, similarly to Sec. II. Note that at zeroth order our solution is again the unstabilized RS1 solution. At first order χ_1 is given again by Eq. (2.19), although A'_1 may now be a non-zero constant. As discussed above, we can solve for the A'_n using Eq. (5.7) in terms of the $\chi_{m<n}$ and $\chi'_{m<n}$. Therefore, A'_2 must be a quadratic polynomial in χ_1 and χ'_1. Summarizing, we have

$$A'_0 + A'_1 + A'_2 = \alpha_0 + \alpha_1 \chi_1 + \alpha_2 \chi'_1 + \frac{1}{12} \epsilon \chi_1^2 - \frac{1}{12} \chi_1'^2,$$
(5.15)

where $\alpha_0 = -1 + \mathcal{O}(\lambda)$, $\alpha_{1,2} \sim \mathcal{O}(\lambda)$. The constants α_i are independent of y_c since the A'_n are determined in terms of the χ_m and χ'_m locally.

It remains to satisfy the junction conditions for Eq. (5.6). After an $\mathcal{O}(\lambda^2)$ tuning of ρ_v (not appearing at higher orders), similar to Sec. II, the above considerations and Eq. (2.19) yield a visible junction condition of the form

$$P_2(e^{\Delta - y_c}) = 0,$$
(5.16)

where P_2 is a quadratic polynomial with order one coefficients. For generic values of these coefficients one obtains solutions,

$$e^{\Delta - y_c} = \mathcal{O}(1).$$
(5.17)

Thus, along with Eq. (5.15) we have demonstrated the hierarchy, Eq. (2.17), at this order. The hidden brane junction condition is solved by fine-tuning ρ_h, corresponding to the fine-tuning of the 4D cosmological constant to zero.

C. Subdominance of higher orders

We claim that at any order χ_n has the general form (2.33), and A'_n has the general form

$$A'_n = \sum q_{n,r,s,t}^{\vec{}} \lambda^{n_1} (\lambda y)^{n_2} (\lambda y_c)^{n_3} (\lambda / \Delta_-)^{n_4}$$

$$\times e^{r\Delta_+(y - y_c)} e^{s\Delta_- y} e^{t\Delta_- y_c},$$
(5.18)

so that perturbation theory is well-behaved. Therefore, our second order mechanism above for stabilization of the weak or Planck hierarchy of order $e^{-\mathcal{O}(1)/\epsilon}$ continues to hold.

We show this claim is true by induction. We have already discussed the cases for $n = 0,1$. Focusing now on χ_n and assuming that Eq. (2.33) is valid at all orders lower than n, the nth order source has the form

$$J_n = \sum a_{n,r,s,t}^{\vec{}} \lambda^{n_1} (\lambda y)^{n_2} (\lambda y_c)^{n_3} (\lambda / \Delta_-)^{n_4} e^{r\Delta_+(y - y_c)}$$

$$e^{s\Delta_- y} e^{t\Delta_- y_c} + \sum b_{n,r,s,t}^{\vec{}} \lambda^{n_1} (\lambda y)^{n_2} (\lambda y_c)^{n_3} (\lambda / \Delta_-)^{n_4}$$

$$\times e^{-r\Delta_+ y_c} e^{t\Delta_- y_c} \delta(y)$$

$$+ \sum c_{n,r,s,t}^{\vec{}} \lambda^{n_1} (\lambda y)^{n_2} (\lambda y_c)^{n_3} (\lambda / \Delta_-)^{n_4}$$

$$\times e^{-r\Delta_+ y_c} e^{t\Delta_- y_c} \delta(y - y_c).$$
(5.19)

It is straightforward to check that with this source, the solution for χ_n given by Eq. (2.36) indeed satisfies Eq. (2.33). Then, given the form of Eq. (5.7) and the form for χ_n of Eqs. (2.33), (5.18) readily follows.

D. The case of $\lambda > \epsilon$

A significant difference between the source considered in this section, Eq. (5.11) and that considered in Sec. II, Eq. (2.32), is that in the latter every power of χ without a derivative was accompanied by an explicit power of $\sqrt{\epsilon}$. Recall that this ensured that increasing powers of $1/\Delta_-$ in perturbation theory were accompanied by ϵ factors. This allowed us to take $\lambda > \epsilon$ and still have a meaningful perturbative expansion. Without this feature, in this section we are limited to formally $\lambda < \epsilon$, as we have assumed thus far in this section, in order to have a good expansion.

However, there is an interesting scenario under which we would recover a good expansion for $\lambda > \epsilon$ even for higher-derivative operators. We first assume a (non-linearly realized) global symmetry,[4]

$$\chi \to \chi + \text{const.}$$
(5.20)

If this symmetry were exact, only derivatives of χ would be permitted in the action. But one does not expect that such global symmetries are respected by the underlying quantum

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[4]We thank W. Goldberger for pointing out the usefulness of this symmetry to us.

gravity theory. We will assume that in fact such violation is controlled by the small parameter, ϵ, so that non-derivative powers of χ in the action are accompanied by $\sqrt{\epsilon}$, thereby generalizing the explicit appearance of ϵ in Sec. II. Then, increasing powers of $1/\Delta_-$ in perturbation theory are again accompanied by ϵ factors, so that the perturbative expansion is under control for $\lambda > \epsilon$.

VI. CONCLUSIONS

We demonstrated the stability of the RS mechanism for generating the weak-Planck hierarchy in the presence of higher-derivative interactions. We re-worked the Goldberger-Wise radius stabilization mechanism in a systematic perturbative expansion in parameters of the brane potential. Incorporating higher-derivative interactions as further perturbations, we showed that they did not affect the basic mechanism.

In our perturbative analysis, radius stabilization is achieved in the following steps. At zeroth order we found the unstabilized RS1 vacuum with a trivial profile for the Goldberger-Wise scalar. At first order, the scalar responds non-trivially to the brane-potentials. The back-reaction of this scalar profile on the 5D metric (warp factor) takes place at second order. In particular, the radius acquires a distinct, stable value. This value can naturally be several times the fundamental scale of the theory, while the bulk geometry is a small perturbation of AdS$_5$, so that the RS mechanism for generating the hierarchy operates. We also gave a careful treatment of all higher orders in perturbation theory, showing that our expansion is under good control.

The derivative expansion of effective field theory seems at first fundamentally at odds with the presence of "thin" or δ-function branes or orbifold fixed points, resulting in ill-defined singularities in the classical equations of motion. While such singularities can be regulated, the process is messy and apparently regulator-dependent. However, we showed how classical renormalization of the brane action can be implemented so as to remove the need for explicit regulators. This greatly simplifies the discussion of higher-derivative effects.

It is important to study the stability of the RS1 effective field theory under quantum effects. Since the effective theory is not renormalizable we expect quantum divergences of the form of every possible local operator (subject only to symmetries). However, once these divergences are covariantly regulated at the fundamental scale, they must be of the form and strength of the higher-derivative interactions already considered in this paper. As usual in effective field theory, renormalization proceeds order by order in the derivative expansion, using counterterms also of the form of the operators of this paper. Thus, because quantum divergences are local they cannot destabilize the RS hierarchy, since our purely classical analysis already treats all such local effects. This leaves only the non-local quantum effects which are UV-finite and therefore well-defined and calculable. In future work, we hope to build on Refs. [6] in studying the general structure of these non-local quantum amplitudes and whether they affect the basic RS1 mechanism.

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